Legendre Spectral Galerkin Method for Two-Dimensional Hammerstein Integral Equations

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Abstract

This paper focuses on the approximate solution of the two dimensional Hammerstein integral equations with smooth kernels. We employ Legendre spectral Galerkin method to approximate the solution of these nonlinear integral equations. We analyze the convergence behaviour of the proposed method under both L^2 and L^∞ norms. We are able to obtain superconvergence rates for Hammerstein integral equations with smooth kernels.

Keywords: *Hammerstein integral equations, Spectral method, Galerkin method, Legendre polynomials, Superconvergence rates*

Introduction

Non-linear Hammerstein integral equations, particularly in two dimensions, are critical in modeling numerous phenomena across diverse scientific and engineering disciplines, ranging from fluid dynamics and quantum mechanics to population dynamics and chemical reactions [1]. A computational approach to solve integral equations is an essential work in scientific research . In general these equations cannot be solved explicitly, so we have several numerical methods to solve them as projection methods, variational iteration methods, degenerate kernel methods, quadrature methods etc.
This paper focuses on deriving superconvergence results for two dimensional non linear Fredholm integral equations of Hammerstein type with smooth kernels. We employ Legendre Galerkin method to approximate the solutions of these integral equations. We analyze the convergence behaviour of the proposed method in both L^2 norm and infinity norm. Use of Legendre polynomial imply smaller nonlinear systems ensuring less expensive computation and it also has an orthogonal property, leading to improved convergence properties.

We prove that the approximated solution of the Legendre Galerkin method converges to the exact solution with the order $\mathcal{O}(\max\{m^{-r_1}, n^{-r_2}\})$ in L^2 and infinity norms, and the iterated Legendre Galerkin solution converges with the order $\mathcal{O}(\max\{m^{-2r_1}, n^{-2r_2}\})$ in both L^2 and infinity norms, m, n being the highest degree of Legendre polynomial employed in the approximation and r_1, r_2 being the smoothness of the kernels.
We organize this paper as follows. In Sect. 2, we discuss the Legendre spectral Galerkin method for the equation of type ((1.1)). In Sect. 3, we obtain the existence of the approximate and iterated approximate solutions. In Sect. 4, we obtain the convergence results for the approximate and iterated approximate solutions.Throughout this paper, we assume that c is a generic constant.
Let $\Omega = [-1,1] \times [-1,1] \subset \mathbb{R}^2$ and $\mathbb{X} = \mathcal{C}(\Omega)$ be the Banach space of continuous functions in Ω. Consider the following two dimensional Hammerstein integral equation

$$u(s,t) = \iint_{\Omega} k(s,t,x,y)\psi(x,y,u(x,y))\,dx\,dy + f(s,t), \quad (s,t) \in \Omega, \tag{1.1}$$

where k , f and ψ are known functions and u is the unknown function to be determined. We take $k(\cdot,\cdot,\cdot,\cdot)$ to be a smooth function.

The following assumptions are made on f, k, and ψ :

1. $f \in \mathcal{C}(\Omega)$.
2. $\lim_{(s,t)\to(s',t')} ||k(s,t,.,.) - k(s',t',.,.)||_\infty = 0 \ \ , (s,t), (s',t') \in \Omega.$
3. $M = ||k||_\infty = \sup_{(s,t),(x,y)\in\Omega} |k(s,t,x,y)| < \infty.$
4. The nonlinear function $\psi(x,y,u)$ is bounded and continuous over $\Omega \times \mathbb{R}^2$. $\psi(x,y,u)$ is Lipschitz continuous in u, i.e., for any $u_1, u_2 \in \mathbb{R}^2$, $\exists \ \ c_1 > 0$ such that
$$|\psi(x,y,u_1) - \psi(x,y,u_2)| \le c_1|u_1 - u_2|, \ \ \forall \ (x,y) \in \Omega.$$
5. The partial derivative $\psi^{(0,0,1)}(x,y,u(x,y))$ of ψ w.r.t the third variable exists and is Lipschitz continuous in u, i.e., for any $u_1, u_2 \in \mathbb{R}^2$, $\exists \ c_2 > 0$ such that
$$|\psi^{(0,0,1)}(x,y,u_1) - \psi^{(0,0,1)}(x,y,u_2)| \le c_2|u_1 - u_2|, \ \forall \ \ (x,y) \in \Omega.$$

Let $C^{(r_1,r_2)}(\Omega)$ denote the set of all continuously differentiable function of order r_1 and r_2 on Ω.

For $k(s,t,x,y) \in C^{(r_1,r_2)}(\Omega) \times C^{(r_1,r_2)}(\Omega)$, we denote

$$D^{(i,i',j,j')}k(s,t,x,y) = \frac{\partial^{i+i'+j+j'}}{\partial s^i \partial t^{i'} \partial x^j \partial y^{j'}} k(s,t,x,y).$$

and we set

$$||k||_{r_1.r_2,\infty} = \sum_{i,j=0}^{r_1} \sum_{i',j'=0}^{r_2} ||D^{(i,i',j,j')} k(s,t,x,y)||_\infty.$$

Let

$$\mathcal{K}y(s,t) = \int\int_\Omega k(s,t,u,v)y(u,v)dudv, \ (s,t) \in \Omega, \ y \in \mathbb{X}.$$

Using Holder's inequality we have for any $y \in \mathbb{X}$,

$$\begin{aligned}
||\mathcal{K}y||_\infty = \sup_{(s,t)\in\Omega} |\mathcal{K}y(s,t)| &= \sup_{(s,t),(u,v)\in\Omega} |\int\int_\Omega k(s,t,u,v)y(u,v)dudv| \\
&\leq \sup_{(s,t),(u,v)\in\Omega} |k(s,t,u,v)| \int\int_\Omega |y(u,v)|dudv \\
&\leq 2M||y||_{L^2}. \qquad (1.2)
\end{aligned}$$

and

$$||\mathcal{K}y||_{L^2} \leq 2||\mathcal{K}y||_\infty \leq 4M||y||_{L^2}. \qquad (1.3)$$

Now using Kumar and Sloan [2] method to find an approximate solution of (1.1), the projection method will be now applied to an equivalent equation for the function z defined by

$$z(s,t) := \psi(s,t,u(s,t)), \ (s,t) \in \Omega. \qquad (1.4)$$

And as $\psi(.,.,.) \in C^{(r_1,r_2)}(\Omega \times \mathbb{R}^2)$ and $u \in C^{(r_1,r_2)}(\Omega)$, we can obtain that $z \in C^{(r_1,r_2)}(\Omega)$. The desired exact solution u of (1.1) is obtained by

$$u(s,t) = f(s,t) + \int_{-1}^1 \int_{-1}^1 k(s,t,x,y)z(x,y)dxdy, \ (s,t) \in \Omega. \qquad (1.5)$$

For our convenience, we consider a nonlinear operator $\Psi : \mathbb{X} \to \mathbb{X}$ defined by

$$\Psi(u)(s,t) := \psi(s,t,u(s,t)). \qquad (1.6)$$

Then (1.1) will take the form

$$u = \mathcal{K}z + f, \qquad (1.7)$$

and (1.6) becomes

$$z = \Psi(\mathcal{K}z + f). \qquad (1.8)$$

Let $\mathcal{T}(v) := \Psi(\mathcal{K}v + f),\ v \in \mathbb{X}$, then the (1.8) can be written as

$$z = \mathcal{T}z. \tag{1.9}$$

Theorem 1.1. *Let $\mathbb{X} = C(\Omega), f \in \mathbb{X}$ and $k(.,.,.,.) \in C(\Omega \times \Omega)$ with $M = \sup_{(s,t),(x,y)\in\Omega} |k(s,t,x,y)| < \infty$. Let $\psi(x,y,u(x,y)) \in C(\Omega \times \mathbb{R}^2)$ satisfies the Lipschitz condition in the third variable, i.e.,*

$$|\psi(x,y,u_1) - \psi(x,y,u_2)| \leq c_1|u_1 - u_2|, \quad u_1, u_2 \in \mathbb{X},$$

with $4Mc_1 < 1$. Then the operator equation $z = \mathcal{T}z$ has a unique solution $z_0 \in \mathbb{X}$ i.e., we have $z_0 = \mathcal{T}z_0$.

Proof. Let $z_1, z_2 \in C(\Omega)$. Using Lipschitz's continuity of $\psi(.,.,u(.,.))$ and the estimate (1.2) , we have

$$\begin{aligned} ||\mathcal{T}z_1 - \mathcal{T}z_2||_\infty &= ||\Psi(\mathcal{K}z_1 + f) - \Psi(\mathcal{K}z_2 + f)||_\infty \\ &\leq c_1||\mathcal{K}(z_1 - z_2)||_\infty \\ &\leq c_1 2M||z_1 - z_2||_{L^2} \\ &\leq 4Mc_1||z_1 - z_2||_\infty. \end{aligned} \tag{1.10}$$

By assumption $4Mc_1 < 1$, hence $\mathcal{T}$ is a contraction mapping on $\mathbb{X}$. Since $\mathbb{X} = C(\Omega)$ with $||.||_\infty$ norm is a Banach space, $\mathcal{T}$ has a unique fixed point in $\mathbb{X}$, by Banach contraction theorem. We denote this unique solution by z_0. Hence the proof follows. □

Legendre Spectral Galerkin Method

To find an approximate solution of the integral equation (1.5) we choose a finite dimensional family of functions which is believed to contain a function close to the true solution. There are various senses in which the approximate solution can be said to satisfy the equation (1.5) and these leads to different types of methods [3] [4]. In this paper we will discuss the Galerkin method, which when formulated in an abstract framework, make use of projection operators.[5]
Let $\mathbb{X}_m = \text{span}\{\phi_i(x)\}_{i=0}^m$ and $\mathbb{X}_n = \text{span}\{\phi_j(x)\}_{j=0}^n$ be the sequence of Legendre polynomial subspaces of $\mathbb{X}$ of degree i and j respectively. Let $\mathbb{X}_{mn} = \mathbb{X}_m \times \mathbb{X}_n = \{\phi_i\phi_j : 0 \leq i \leq m, 0 \leq j \leq n\}$ is the set of all polynomials of degree atmost mn on Ω. And the set $\{\phi_i\phi_j : 0 \leq i \leq m, 0 \leq j \leq n\}$ forms an orthonormal basis for $\mathbb{X}_{mn} \subset \mathbb{X}$. For any non-negative integer $r_1, r_2 \in \mathbb{N}$, define

$$H_2^{r_1,r_2}(\Omega) = \{u(x,y)|\frac{\partial^{i+j}}{\partial x^i \partial y^j}u(x,y) \in L^2(\Omega), |i| \leq r_1, |j| \leq r_2, |i+j| \leq \max\{r_1, r_2\}\}.$$

Let $\mathbb{X} = C([-1,1]\times[-1,1])$ and $\mathcal{P}_m : \mathbb{X} \to \mathbb{X}_m$ be the Legendre orthogonal projection operator defined by

$$\mathcal{P}_m u(s,t) = \sum_{i=0}^{m} < u(.,t), \phi_i > \phi_i(s), u \in \mathbb{X},$$

where $< u(.,t), \phi_i >= \int_{-1}^{1} u(\xi,t)\phi_i(\xi)d\xi$ for $i = 0,1,2,...,m$ and for fixed $t \in [-1,1]$. Similarly let $\mathcal{P}_n : \mathbb{X} \to \mathbb{X}_n$ be the Legendre orthogonal projection operator defined by

$$\mathcal{P}_n u(s,t) = \sum_{j=0}^{n} < u(s,.), \phi_j > \phi_j(t), u \in \mathbb{X},$$

where $< u(s,.), \phi_j >= \int_{-1}^{1} u(s,\eta)\phi_j(\eta)d\eta$ for $j = 0,1,2,...,n$ and for fixed $s \in [-1,1]$.
Orthogonal projection operator : We define the orthogonal projection operator $\mathcal{P}_{mn} : \mathbb{X} \to \mathbb{X}_{mn}$ by

$$\mathcal{P}_{mn} u(s,t) = \sum_{i=0}^{m}\sum_{j=0}^{n} \langle u, \phi_i\phi_j \rangle \, \phi_i(s)\phi_j(t), \quad u \in \mathbb{X}, \tag{2.1}$$

where
$$\langle u, \phi_i\phi_j \rangle = \int_{-1}^{1}\int_{-1}^{1} u(x,y)\phi_i(x)\phi_j(y)\, dxdy.$$

Also from [6] we obtained that $\mathcal{P}_m\mathcal{P}_n u(s,t) = \mathcal{P}_{mn} u(s,t)$.

Lemma 2.1. *For fixed* $t \in [-1,1]$, $u(s,t) \in H_2^{r_1,0}(\Omega)$ *and* $r_1 \geq 0$

$$||u - \mathcal{P}_m u||_{L^2} \leq c m^{-r_1} ||u||_{r_1,0,2},$$
$$||u - \mathcal{P}_m u||_{\infty} \leq c m^{\frac{3}{4}-r_1} ||u||_{r_1,0,2},$$

where c is a constant independent of m.

Lemma 2.2. *For fixed* $s \in [-1,1]$, $u(s,t) \in H_2^{0,r_2}(\Omega)$ *and* $r_2 \geq 0$

$$||u - \mathcal{P}_n u||_{L^2} \leq c n^{-r_2} ||u||_{0,r_2,2},$$
$$||u - \mathcal{P}_n u||_{\infty} \leq c n^{\frac{3}{4}-r_2} ||u||_{0,r_2,2},$$

where c is a constant independent of n.

Theorem 2.1. *[7]For any* $u \in H_2^{r_1,r_2}(\Omega)$, *the following results hold:*

$$||u - \mathcal{P}_{mn} u||_{L^2} \leq c \max\{m^{-r_1}, n^{-r_2}\} ||u||_{r_1,r_2,2},$$
$$||u - \mathcal{P}_{mn} u||_{\infty} \leq c \max\{m^{\frac{3}{4}-r_1}, n^{\frac{3}{4}-r_2}\} ||u||_{r_1,r_2,2}.$$

Proof. By using Lemma (2.1), Lemma (2.2) and $\mathcal{P}_{mn} = \mathcal{P}_m\mathcal{P}_n$, we obtain

$$\begin{aligned} ||u - \mathcal{P}_{mn}u||_{L^2} =& ||u - \mathcal{P}_m u + \mathcal{P}_m u - \mathcal{P}_{mn}u||_{L^2} \leq ||u - \mathcal{P}_m u||_{L^2} + ||\mathcal{P}_m u - \mathcal{P}_{mn}u||_{L^2} \\ &= ||u - \mathcal{P}_m u||_{L^2} + ||\mathcal{P}_m||_{L^2}||u - \mathcal{P}_n u||_{L^2} \leq cm^{-r_1}||u||_{r_1,0,2} + cn^{-r_2}||u||_{0,r_2,2} \\ &\leq c\max\{m^{-r_1}, n^{-r_2}\}||u||_{r_1,r_2,2}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} ||u - \mathcal{P}_{mn}u||_\infty =& ||u - \mathcal{P}_m u + \mathcal{P}_m u - \mathcal{P}_{mn}u||_\infty \leq ||u - \mathcal{P}_m u||_\infty + ||\mathcal{P}_m u - \mathcal{P}_{mn}u||_\infty \\ &= ||u - \mathcal{P}_m u||_\infty + ||\mathcal{P}_m||_\infty||u - \mathcal{P}_n u||_\infty \leq cm^{\frac{3}{4}-r_1}||u||_{r_1,0,2} + cn^{\frac{3}{4}-r_2}||u||_{0,r_2,2} \\ &\leq c\max\{m^{\frac{3}{4}-r_1}, n^{\frac{3}{4}-r_2}\}||u||_{r_1,r_2,2}. \end{aligned} \tag{2.3}$$

□

So, we have

$$||u - \mathcal{P}_{mn}u||_{L^2} \leq c\max\{m^{-r_1}, n^{-r_2}\}||u||_{r_1,r_2,2}, \tag{2.4}$$

$$||u - \mathcal{P}_{mn}u||_\infty \leq c\max\{m^{\frac{3}{4}-r_1}, n^{\frac{3}{4}-r_2}\}||u||_{r_1,r_2,2}, \tag{2.5}$$

where c is a constant independent of m, n. Also from the discussed properties of operator, we have

$$||\mathcal{P}_n u||_{L^2} \leq p||u||_\infty, \quad u \in \mathbb{X}, \tag{2.6}$$

where p is a constant independent of m, n. The projection method for $z = \Psi(\mathcal{K}z + f)$ is seeking an approximate solution $z_{mn} \in \mathbb{X}_{mn}$ such that

$$z_{mn} = \mathcal{P}_{mn}\Psi(\mathcal{K}z_{mn} + f). \tag{2.7}$$

Let $\mathcal{T}_{mn}$ be the operator defined by

$$\mathcal{T}_{mn}(v) := \mathcal{P}_{mn}\Psi(\mathcal{K}v + f), \quad u \in \mathbb{X}.$$

Then (2.7) can be written as

$$z_{mn} = \mathcal{T}_{mn}z_{mn}. \tag{2.8}$$

Corresponding approximate solution u_{mn} of u is given by

$$u_{mn} = \mathcal{K}z_{mn} + f. \tag{2.9}$$

In order to obtain more accurate approximation solution, we further consider the iterated projection method for $z = \Psi(\mathcal{K}z + f)$ and see that the iterated solution $\tilde{z}_{mn}$ satisfies the following equation

$$\tilde{z}_{mn} = \Psi\left(\mathcal{K}\mathcal{P}_{mn}\tilde{z}_{mn} + f\right). \tag{2.10}$$

Letting $\tilde{\mathcal{T}}_{mn}(v) := \Psi(\mathcal{K}\mathcal{P}_{mn}v + f),\ v \in \mathbb{X}$, the above equation can be written as

$$\tilde{z}_{mn} = \tilde{\mathcal{T}}_{mn}\tilde{z}_{mn}. \tag{2.11}$$

Corresponding approximate solution $\tilde{u}_{mn}$ of u is given by

$$\tilde{u}_{mn} = \mathcal{K}\tilde{z}_{mn} + f. \tag{2.12}$$

Existence of approximate and iterated approximate solutions and their error bounds

To discuss the existence of approximate solutions we recall the following definition of ν-convergence and a lemma from [8]

Definition 3.1. *Let $\mathbb{X}$ be a Banach space and $\mathbb{BL}(\mathbb{X})$ be space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$. Let $\mathcal{F}_n, \mathcal{F} \in \mathbb{BL}(\mathbb{X})$. We say $\mathcal{F}_n$ is $\nu-$ convergent to $\mathcal{F}$ if*

$$||\mathcal{F}_n|| \leq c < \infty, ||(\mathcal{F}_n - \mathcal{F})\mathcal{F}|| \to 0, ||(\mathcal{F}_n - \mathcal{F})\mathcal{F}_n|| \to 0,\ as\ n \to \infty.$$

Lemma 3.1. *Let $\mathbb{X}$ be a Banach space and $\mathcal{F}$, $\mathcal{F}_n$ be bounded linear operators on $\mathbb{X}$. If $\mathcal{F}_n$ is norm convergent or ν-convergent to $\mathcal{F}$ and $(I - \mathcal{F})^{-1}$ exists, then $(I - \mathcal{F}_n)^{-1}$ exists and is uniformly bounded on $\mathbb{X}$.*

Lemma 3.2. *Let $\mathcal{T}'(z_0)$ and $\tilde{\mathcal{T}}'_{mn}(z_0)$ be the Frechet derivatives of $\mathcal{T}(z)$ and $\tilde{\mathcal{T}}_{mn}(z)$, respectively at z_0. Then*

$$\begin{aligned} ||(\mathcal{I} - \mathcal{P}_{mn})\tilde{\mathcal{T}}'_{mn}(z_0)||_{L^2} \to 0, as\ m,\ n \to \infty, \\ ||(\mathcal{I} - \mathcal{P}_{mn})\mathcal{T}'(z_0)||_{L^2} \to 0, as\ m,\ n \to \infty. \end{aligned}$$

Proof. We have $\tilde{\mathcal{T}}_{mn}(z_0) = \Psi'(\mathcal{K}\mathcal{P}_{mn}z_0 + f)\mathcal{K}\mathcal{P}_{mn}$ Now using the Lipschitz's continuity of $\psi^{(0,0,1)}(.,.,u(.,.))$ and boundedness of $||\Psi'(\mathcal{K}z_0 + f)||_\infty$, we have

$$\begin{aligned} ||\Psi'(\mathcal{K}\mathcal{P}_{mn}z_0 + f)||_\infty &\leq ||\Psi'(\mathcal{K}\mathcal{P}_{mn}z_0 + f) - \Psi'(\mathcal{K}z_0 + f)||_\infty + ||\Psi'(\mathcal{K}z_0 + f)||_\infty \\ &\leq c_2||\mathcal{K}(\mathcal{P}_{mn} - \mathcal{I})z_0||_\infty + ||\Psi'(\mathcal{K}z_0 + f)||_\infty. \end{aligned} \tag{3.1}$$

Now

$$||\mathcal{K}(\mathcal{I}-\mathcal{P}_{mn})z_0||_\infty = \sup_{s,t\in\Omega}|\mathcal{K}(\mathcal{I}-\mathcal{P}_{mn})z_0(s,t)| = \sup_{s,t\in\Omega}|\int_{-1}^{1}\int_{-1}^{1}k(s,t,x,y)(\mathcal{I}-\mathcal{P}_{mn})z_0(x,y)dxdy|$$
$$= \sup_{s,t\in\Omega}| < k_{s,t}(.,.),(\mathcal{I}-\mathcal{P}_{mn})z_0 > | \leq \sup_{s,t\in\Omega}||k_{s,t}(.,.)||_{L^2}||(\mathcal{I}-\mathcal{P}_{mn})z_0||_{L^2}$$
$$\leq 2M||(\mathcal{I}-\mathcal{P}_{mn})z_0||_{L^2} \leq 2cM\max\{m^{-r_1},n^{-r_2}\}||z_0||_{r_1,r_2,2} \to 0 \;\; as \;\; m,n\to\infty. \tag{3.2}$$

Using (3.2) in (3.1) we get

$$||\Psi'(\mathcal{K}\mathcal{P}_{mn}z_0+f)||_\infty \leq B < \infty, \tag{3.3}$$

where B is constant independent of m,n.
Also we have

$$||\Psi'(\mathcal{K}\mathcal{P}_{mn}z_0+f)||_{L^2} \leq 2||\Psi'(\mathcal{K}\mathcal{P}_{mn}z_0+f)||_\infty \leq 2B < \infty. \tag{3.4}$$

Next, let $\bar{B} := \{x \in \mathbb{X} : ||x||_{L^2} \leq 1\}$ be the closed unit ball in $\mathbb{X}$. We have $\tilde{\mathcal{T}}'_{mn}(z_0) = \Psi'(\mathcal{K}\mathcal{P}_{mn}z_0+f)\mathcal{K}\mathcal{P}_{mn}$. Since $\{\mathcal{K}\mathcal{P}_{mn}\}$ is a sequence of compact operators and $\Psi'(\mathcal{K}\mathcal{P}_{mn}z_0+f)$ is uniformly bounded. $\tilde{\mathcal{T}}'_{mn}(z_0)$ are compact operators. Thus $S = \{\tilde{\mathcal{T}}'_{mn}(z_0)x : x \in \bar{B}, m,n \in \mathbb{N}\}$ is relatively compact set. So, we can conclude that

$$||(\mathcal{I}-\mathcal{P}_{mn})\tilde{\mathcal{T}}'_{mn}(z_0)||_{L^2} = \sup\{||(\mathcal{I}-\mathcal{P}_{mn})\tilde{\mathcal{T}}'_{mn}(z_0)x||_{L^2} : x\in\bar{B}\}$$
$$= \sup\{||(\mathcal{I}-\mathcal{P}_{mn})y||_{L^2} : y \in S\} \to 0 \;\; as \;\; m,n\to\infty. \tag{3.5}$$

Similarly, since $\Psi'(\mathcal{K}z_0+f)$ is bounded and $\mathcal{K}$ is compact, $\tilde{\mathcal{T}}'(z_0) = \Psi'(\mathcal{K}z_0+f)\mathcal{K}$ is also compact and we have $||(\mathcal{I}-\mathcal{P}_{mn})\mathcal{T}'(z_0)||_{L^2} \to 0 \;\; as \;\; m,n\to\infty$. □

Vainikko's theorem from [9] gives us the condition under which the solvability of one equation leads to the solvability of other equation, so we have following theorems :

Theorem 3.2. *Let $z_0 \in C^{(r_1,r_2)}([-1,1]\times[-1,1])$ be an isolated solution of $z = \Psi(\mathcal{K}z+f)$. Assume that 1 is not an eigenvalue of $\Psi'(\mathcal{K}z_0+f))\mathcal{K}$. Let $\mathcal{P}_{mn} : \mathbb{X} \to \mathbb{X}_{mn}$ be an orthogonal projection operator defined by (2.1). Then the eq $z_{mn} = \mathcal{P}_{mn}\Psi(\mathcal{K}z_{mn}+f)$ has a unique solution $z_{mn} \in B(z_0,\delta) = \{z : ||z-z_0||_{L^2} < \delta\}$ for some $\delta > 0$ and for sufficiently large m,n. Moreover, there exists a constant $0 < q < 1$, independent of m,n such that*

$$\frac{\alpha_N}{1+q} \leq ||z_{mn}-z_0||_{L^2} \leq \frac{\alpha_N}{1-q},$$

where $\alpha_N = ||(I - \mathcal{T}'_{mn}(z_0))^{-1}(\mathcal{T}_{mn}(z_0) - \mathcal{T}(z_0))||_{L^2}$*. Further, we obtain*

$$||z_{mn} - z_0||_{L^2} \leq c||(\mathcal{P}_{mn} - I)z_0||_{L^2} = \mathcal{O}(max\{m^{-r_1}, n^{-r_2}\}),$$

where c *is a constant independent of* m, n.

Proof. Using Lemma (3.2), we have

$$\begin{aligned} ||\mathcal{T}'_{mn}(z_0) - \mathcal{T}'(z_0)||_{L^2} &= ||\mathcal{P}_{mn}\Psi'(\mathcal{K}z_0 + f)\mathcal{K} - \Psi'(\mathcal{K}z_0 + f)\mathcal{K})||_{L^2} \\ &= ||(\mathcal{P}_{mn} - \mathcal{I})\Psi'(\mathcal{K}z_0 + f)\mathcal{K}||_{L^2} \\ &= ||(\mathcal{P}_{mn} - \mathcal{I})\mathcal{T}'(z_0))||_{L^2} \to 0, as\ m,\ n \to \infty. \end{aligned}$$

Since we assume that 1 is not an eigen value of $\mathcal{T}'(z_0)$, $(\mathcal{I} - \mathcal{T}'(z_0))$ is invertible. Hence by applying Lemma (3.1), we have $(\mathcal{I} - \mathcal{T}'_{mn}(z_0))^{-1}$ exists and uniformly bounded on $\mathbb{X}$, for some sufficiently large m, n, i.e., there exists some $A_1 > 0$ such that $||(\mathcal{I} - \mathcal{T}'_{mn}(z_0))^{-1}||_{L^2} \leq A_1 < \infty$. Now from estimates (1.2) and (2.6), we have for any $z \in B(z_0, \delta)$,

$$\begin{aligned} ||[\mathcal{T}'_{mn}(z_0) - \mathcal{T}'_{mn}(z)]v||_{L^2} &= ||[\mathcal{P}_{mn}\Psi'(\mathcal{K}z_0 + f)\mathcal{K} - \mathcal{P}_{mn}\Psi'(\mathcal{K}z + f)\mathcal{K}]v||_{L^2} \\ &\leq p||[\Psi'(\mathcal{K}z_0 + f) - \Psi'(\mathcal{K}z + f)]\mathcal{K}v||_\infty \\ &\leq p||\Psi'(\mathcal{K}z_0 + f) - \Psi'(\mathcal{K}z + f)||_\infty||\mathcal{K}v||_\infty \\ &\leq 2pM||\Psi'(\mathcal{K}z_0 + f) - \Psi'(\mathcal{K}z + f)||_\infty||v||_{L^2}. \end{aligned} \tag{3.6}$$

Taking use of the Lipschtiz's continuity of $\psi^{(0,0,1)}(., ., u(., .))$ and $||\mathcal{K}y||_\infty \leq 2M||y||_{L^2}$ we have

$$||\Psi'(\mathcal{K}z_0 + f) - \Psi'(\mathcal{K}z + f)||_\infty \leq c_2||\mathcal{K}(z_0 - z)||_\infty \leq 2c_2M||z_0 - z||_{L^2} \leq 2Mc_2\delta. \tag{3.7}$$

Using the estimate (3.7) in (3.6), we obtain

$$||[\mathcal{T}'_{mn}(z_0) - \mathcal{T}'_{mn}(z)]v||_{L^2} \leq 4pM^2c_2\delta||v||_{L^2}.$$

thus we have

$$\sup_{||z - z_0||_{L^2} \leq \delta} ||(\mathcal{I} - \mathcal{T}'_{mn}(z_0))^{-1}(\mathcal{T}'_{mn}(z_0) - \mathcal{T}'_{mn}(z))||_{L^2} \leq 4pA_1M^2c_2\delta \leq q(say).$$

Here we choose δ in such a way that, $0 < q < 1$. This proves the Eq. (2.35) of Theorem

2.2 of [10] and we have

$$\begin{aligned}\alpha_N =&||(\mathcal{I}-\mathcal{T}'_{mn}(z_0))^{-1}(\mathcal{T}_{mn}(z_0)-\mathcal{T}(z_0))||_{L^2} \leq A_1||\mathcal{T}_{mn}(z_0)-\mathcal{T}(z_0)||_{L^2}\\ &= A_1||\mathcal{P}_{mn}\Psi(\mathcal{K}z_0+f)-\Psi(\mathcal{K}z_0+f)||_{L^2} = A_1||(\mathcal{P}_{mn}-\mathcal{I})\Psi(\mathcal{K}z_0+f)||_{L^2}\\ &= A_1||(\mathcal{P}_{mn}-\mathcal{I})z_0||_{L^2} \to 0, as\ m,n\to\infty.\end{aligned}$$

By choosing m,n large enough such that $\alpha_N \leq \delta(1-q)$, the Eq. (2.36) of Theorem 2.2 [10] is satisfied. Hence by applying Theorem 2.2, we obtain

$$\frac{\alpha_N}{1+q} \leq ||z_{mn}-z_0||_{L^2} \leq \frac{\alpha_N}{1-q},$$

and

$$||z_{mn}-z_0||_{L^2} \leq \frac{\alpha_N}{1-q} \leq c||(\mathcal{P}_{mn}-\mathcal{I})z_0||_{L^2}.$$

Hence from estimates (2.4), we have

$$||z_{mn}-z_0||_{L^2} = \mathcal{O}(\max\{m^{-r_1}, n^{-r_2}\}).$$

This completes the proof. □

Next we discuss the existence and convergence of the iterated approximate solution $\tilde{z}_{mn}$ to z_0.

Theorem 3.3. *Let $z_0 \in C^{(r_1,r_2)}([-1,1]\times[-1,1])$ be an isolated solution of $z=\Psi(\mathcal{K}z+f)$. Assume that 1 is not an eigenvalue of $\Psi'(\mathcal{K}z_0+f))\mathcal{K}$. Let $\mathcal{P}_{mn}:\mathbb{X}\to\mathbb{X}_{mn}$ be an orthogonal projection operator defined by eq (2.1). Then the eq $\tilde{z}_{mn}=\Psi(\mathcal{K}\mathcal{P}_{mn}\tilde{z}_{mn}+f)$ has a unique solution in the sphere $B(z_0,\delta)=\{z:||z-z_0||_\infty<\delta\}$ for some $\delta>0$. Moreover, there exists a constant $0<q<1$, independent of m,n such that*

$$\frac{\beta_N}{1+q} \leq ||\tilde{z}_{mn}-z_0||_\infty \leq \frac{\beta_N}{1-q},$$

where

$$\beta_N = ||(I-\tilde{\mathcal{T}}'_{mn}(z_0))^{-1}(\tilde{\mathcal{T}}_{mn}(z_0)-\mathcal{T}(z_0))||_\infty.$$

Further, we obtain

$$||\tilde{z}_{mn}-z_0||_\infty \leq c\sup_{(s,t)\in\Omega}|<k_{s,t},(I-\mathcal{P}_{mn})z_0>|,$$

where c is constant independent of m,n.

Proof. From Theorem 2.5 of [10] we can say, there exists a constant $L > 0$ such that $||(\mathcal{I} - \tilde{\mathcal{T}}'_{mn}(z_0))^{-1}||_\infty \leq L$, for sufficiently large value of m, n.
Consider for any $z \in B(z_0, \delta)$,

$$\begin{aligned} ||[\tilde{\mathcal{T}}'_{mn}(z) - \tilde{\mathcal{T}}'_{mn}(z_0)]v||_\infty =& ||[\{\Psi'(\mathcal{K}\mathcal{P}_{mn}z + f) - \Psi'(\mathcal{K}\mathcal{P}_{mn}z_0 + f)\}\mathcal{K}\mathcal{P}_{mn}]v)]||_\infty \\ &\leq ||\Psi'(\mathcal{K}\mathcal{P}_{mn}z + f) - \Psi'(\mathcal{K}\mathcal{P}_{mn}z_0 + f)||_\infty ||\mathcal{K}\mathcal{P}_{mn}v||_\infty. \end{aligned} \tag{3.8}$$

Using Cauchy-Schwarz inequality and estimates (1.2) and (2.6), we have

$$\begin{aligned} ||\Psi'(\mathcal{K}\mathcal{P}_{mn}z + f) - \Psi'(\mathcal{K}\mathcal{P}_{mn}z_0 + f)||_\infty \leq c_2||\mathcal{K}\mathcal{P}_{mn}(z - z_0)||_\infty \leq 2Mc_2p||z - z_0||_\infty \\ \leq 2Mc_2p\delta. \end{aligned} \tag{3.9}$$

Combining estimates (1.2),(3.8),(3.9), we obtain

$$||[\tilde{\mathcal{T}}'_{mn}(z) - \tilde{\mathcal{T}}'_{mn}(z_0)]v||_\infty \leq 4M^2c_2p^2\delta||v||_\infty. \tag{3.10}$$

This implies

$$\sup_{||z - z_0||_\infty \leq \delta} ||(\mathcal{I} - \tilde{\mathcal{T}}'_{mn}(z_0))^{-1}(\tilde{\mathcal{T}}'_{mn}(z) - \tilde{\mathcal{T}}'_{mn}(z_0))||_\infty \leq 4LM^2c_2p^2\delta \leq q(say),$$

and

$$\begin{aligned} ||\tilde{\mathcal{T}}_{mn}(z_0) - \mathcal{T}(z_0)||_\infty &\leq ||\Psi(\mathcal{K}\mathcal{P}_{mn}z_0 + f) - \Psi(\mathcal{K}z_0 + f)||_\infty \\ &\leq c_1||\mathcal{K}(\mathcal{I} - \mathcal{P}_{mn})z_0||_\infty \to 0 \ as \ n \to \infty. \end{aligned} \tag{3.11}$$

Hence

$$\beta_N = ||(\mathcal{I} - \tilde{\mathcal{T}}'_{mn}(z_0))^{-1}(\tilde{\mathcal{T}}_{mn}(z_0) - \mathcal{T}(z_0))||_\infty \leq Lc_1||\mathcal{K}(\mathcal{I} - \mathcal{P}_{mn})z_0||_\infty \to 0, as\ m,\ n \to \infty.$$

Choose m, n large enough such that $\beta_N \leq \delta(1 - q)$. Thus we obtain

$$\frac{\beta_N}{1 + q} \leq ||\tilde{z}_{mn} - z_0||_\infty \leq \frac{\beta_N}{1 - q},$$

where

$$\beta_N = ||(\mathcal{I} - \tilde{\mathcal{T}}'_{mn}(z_0))^{-1}(\tilde{\mathcal{T}}_{mn}(z_0) - \mathcal{T}(z_0))||_\infty.$$

Thus

$$||\tilde{z}_{mn} - z_0||_\infty \leq \frac{\beta_N}{1-q} \leq c||(\mathcal{I} - \tilde{\mathcal{T}}'_{mn}(z_0))^{-1}(\tilde{\mathcal{T}}_{mn}(z_0) - \mathcal{T}(z_0))$$
$$\leq cL||\Psi(\mathcal{K}\mathcal{P}_{mn}z_0 + f) - \Psi(\mathcal{K}z_0 + f)||_\infty$$
$$\leq cLc_1||\mathcal{K}(P_{mn} - \mathcal{I})z_0||_\infty$$
$$\leq c \sup_{s,t\in\Omega} | < k_{s,t}(.,.), (I - P_{mn})z_0 > |.$$

This completes the proof □

Convergence Analysis

Theorem 4.1. *Let* $u_0 \in C^{(r_1,r_2)}(\Omega)$ *be an isolated solution of the equation* (1.1) *and* u_{mn} *be the Legendre Galerkin approximation of* u_0. *Then there hold*

$$||u_0 - u_{mn}||_{L^2} = \mathcal{O}(max\{m^{-r_1}, n^{-r_2}\}),$$

$$||u_0 - u_{mn}||_\infty = \mathcal{O}(max\{m^{-r_1}, n^{-r_2}\}).$$

Proof. Using estimates (1.2),(1.7),(2.9) and Theorem (3.2) , we have

$$||u_0 - u_{mn}||_\infty = ||\mathcal{K}(z_0 - z_{mn})||_\infty \leq 2M||z_0 - z_{mn}||_{L^2} = \mathcal{O}(max\{m^{-r_1}, n^{-r_2}\}),$$

and

$$||u_0 - u_{mn}||_{L^2} \leq 2||u_0 - u_{mn}||_\infty = \mathcal{O}(\max\{m^{-r_1}, n^{-r_2}\}).$$

Hence the proof follows. □

Theorem 4.2. *Let* $u_0 \in C^{(r_1,r_2)}(\Omega)$ *be an isolated solution of the equation* (1.1) *and* $\tilde{u}_{mn}$ *be the iterated Legendre Galerkin approximation of* u_0. *Then the following superconvergence rates hold*

$$||u_0 - \tilde{u}_{mn}||_{L^2} = \mathcal{O}(max\{m^{-2r_1}, n^{-2r_2}\}),$$
$$||u_0 - \tilde{u}_{mn}||_\infty = \mathcal{O}(max\{m^{-2r_1}, n^{-2r_2}\}).$$

Proof. We have

$$||\tilde{z}_{mn} - z_0||_\infty \leq c \sup_{s,t\in\Omega} | < k_{s,t}(.,.), (\mathcal{I} - \mathcal{P}_{mn})z_0(.,.) > |.$$

Using the orthogonality of the projection operator $\mathcal{P}_{mn}$, Cauchy-Schwarz inequality and

Theorem (2.1) we obtain

$$\begin{aligned}
| < k_{s,t}(.,.), (\mathcal{I} - \mathcal{P}_{mn})z_0(.,.) > | =& | < (\mathcal{I} - \mathcal{P}_{mn})k_{s,t}, (\mathcal{I} - \mathcal{P}_{mn})z_0 > | \\
&\leq ||(\mathcal{I} - \mathcal{P}_{mn})k_{s,t}||_{L^2} ||(\mathcal{I} - \mathcal{P}_{mn})z_0||_{L^2} \\
&\leq c \max\{m^{-2r_1}, n^{-2r_2}\} ||z_0||_{r_1,r_2,2} ||k_{s,t}(.,.)||_{L^2} \\
&\leq c \max\{m^{-2r_1}, n^{-2r_2}\} ||z_0||_{r_1,r_2,2} ||k||_{r_1,r_2,\infty}. \qquad (4.1)
\end{aligned}$$

Hence

$$||\tilde{z}_{mn} - z_0||_\infty \leq c \max\{m^{-2r_1}, n^{-2r_2}\} ||z_0||_{r_1,r_2,2} ||k||_{r_1,r_2,\infty} = \mathcal{O}(\max\{m^{-2r_1}, n^{-2r_2}\}), \qquad (4.2)$$

and

$$||\tilde{z}_{mn} - z_0||_{L^2} \leq 2||\tilde{z}_{mn} - z_0||_\infty = \mathcal{O}(\max\{m^{-2r_1}, n^{-2r_2}\}). \qquad (4.3)$$

Using estimates (1.2), (1.7), (2.12) , we have

$$||u_0 - \tilde{u}_{mn}||_\infty = ||\mathcal{K}(z_0 - \tilde{z}_{mn})||_\infty \leq 2M||\tilde{z}_{mn} - z_0||_{L^2} = \mathcal{O}(\max\{m^{-2r_1}, n^{-2r_2}\}),$$

and

$$||u_0 - \tilde{u}_{mn}||_{L^2} \leq 2||u_0 - \tilde{u}_{mn}||_\infty = \mathcal{O}(\max\{m^{-2r_1}, n^{-2r_2}\}).$$

Hence the proof follows. □

Conclusion

We obtain superconvergence rates for iterated Galerkin approximate solution for Hammerstein integral equations with smooth kernels. In summary, this paper provides an analysis and computational demonstration of how Legendre spectral Galerkin method can effectively approximate and accelerate convergence for nonlinear integral equations in two dimensions, thereby contributing valuable insights to the numerical analysis of integral equations.

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